Design and implementation of model predictive control for a three-tank hybrid system

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Abstract— Hybrid systems are a special class of systems, new enough to have different definitions depending on the approach of the researchers. In this paper, systems which exhibit both continuous and discrete dynamics is considered as hybrid systems.

Importance of this class of systems nowadays is considerable, given the fact that digital machines like computers often have to work in conjunction with analog systems. General method of analysis and design of hybrid systems have not been presented yet, but literature provides many different methods, depending on the character of the system and aim of control.

In this paper a three tank hybrid system is modeled. A model predictive control is designed and implemented for the hybrid model.

Keywords— Hybrid system, model predictive control, three-tank system, hybrid modeling.

I. INTRODUCTION

Hybrid systems are dynamical systems that involve the interaction of different types of dynamics. In this class we are interested in hybrid dynamics that arise out of the interaction of continuous state dynamics and discrete state dynamics. A state variable is called discrete if it takes on a finite (or countable) number of values and continuous if it takes values in Euclidean space \( \mathbb{R}^n \) for some \( n \geq 1 \). By their nature, discrete states can change value only through a discrete “jump. Continuous states can change values either through a jump, or by “flowing” in continuous time according to a differential equation. Hybrid systems involve both these types of dynamics: discrete jumps and continuous flows. The analysis and design of hybrid systems is in general more difficult than that of purely discrete or purely continuous systems, because the discrete dynamics may affect the continuous evolution and vice versa.

The term predictive control refers to a wide range of control methods which make use of a model of the system to compute the control signal by minimizing an objective function. The model is necessary to calculate, at a given instant of time, for which possible future value of the control signal, the cost function value which would be smallest. Usually, the cost function depends on the change in the value of the control signal and difference between the set-point and output value. MPC can be also used for control of hybrid systems; however their digital part must be included. This can be realized by computing the cost function for different configurations of discrete values and choosing one assuring the smallest result. The algorithm used in this paper is presented in Fig. 1..

![Fig. 1 MPC for Hybrid Systems.](image)

II. THE HYBRID THREE TANK SYSTEM

A. Physical system

Fig. 2 shows the construction of three tank system the system consists of three cylindrical tanks of equal dimensions. The tanks are connected to each other through a manual valve. Three rotameter is fixed at the inlet of each tank to measure the flow rate. Each tank is provided with a Differential Pressure Transmitter (DPT) which measures pressure of the liquid column. This information is converted to
water level. To enable various system configurations, vertical tanks are interconnected by valves, which allows us to work with one, two or three tanks, at a time. The system can be configured as interacting as well as non-interacting system.

**B. Hybrid modeling**

![Three tank modeling](image)

The dynamic behavior of the system can be given by the following equations.

\[
\begin{align*}
\frac{dh_1}{dt} &= \frac{q_1}{S_1} - \text{sign}(h_1 - h_3)V_{13}\frac{S_{13}}{S_1}\sqrt{2g(h_1 - h_3)} - V_1\frac{S_{oi}}{S_1}\sqrt{2gh_1} \\
\frac{dh_2}{dt} &= \frac{q_2}{S_2} - \text{sign}(h_2 - h_3)V_{23}\frac{S_{23}}{S_2}\sqrt{2g(h_2 - h_3)} - V_2\frac{S_{oi}}{S_2}\sqrt{2gh_2} \\
\frac{dh_3}{dt} &= \text{sign}(h_1 - h_3)V_{13}\frac{S_{13}}{S_3}\sqrt{2g(h_1 - h_3)} + \text{sign}(h_2 - h_3)V_{23}\frac{S_{23}}{S_3}\sqrt{2g(h_2 - h_3)} - V_3\frac{S_{oi}}{S_3}\sqrt{2gh_3}
\end{align*}
\]

(1)

(2)

(3)

where:

- \(h_1, h_2, h_3\) – levels in respective tanks
- \(S_1, S_2, S_3\) – cross sections of tanks (tanks dimensions are equal)
- \(S_{13}, S_{23}\) – cross section of digital valves between tanks
- \(S_{oi}\) – cross section of output valves
- \(q_1, q_2\) – inflow through pumps
- \(V_{12}, V_{23}\) – status of digital valves between tanks (0-closed, 1-opened)
- \(V_{12}, V_{23}\) – status of output valves (0-closed, 1-opened)
- \(g\) – gravitational acceleration

Given that for the system at hand \(S=S_1=S_2=S_3\), \(S_{oi}=S_{oi}\), \(i=1,2\) and for particular problems, the equations become simpler. They are later on linearized to serve in implementation of the chosen MPC algorithm for the hybrid systems.

Basic parameters of the system are:

- \(a=0.304\text{m}\) – diameter of tanks
- \(v=0.015\text{m}\) – diameter of digital valves
- \(v_3=0.12\text{m}\) – diameter of output valves

- \(L=0.28\text{m}\) – maximum water level in each tank
- \(T_s=0.01\) – variable sampling time.

**III. CONTROL PROBLEM**

It is required that the level in the third tank be constant \(h_3=100\text{mm}\), while all three output valves \(v_1, v_2\) and \(v_3\) are kept open. Both input pumps are working.

The dynamics of such a three tank configuration is given by equations (4), (5) an (6).

\[
\begin{align*}
\frac{dh_1}{dt} &= q_1 - V_{13}\frac{S_{13}}{S_1}\sqrt{2g(h_1 - h_3)} - V_1\frac{S_{oi}}{S_1}\sqrt{2gh_1} \\
\frac{dh_2}{dt} &= q_2 - V_{23}\frac{S_{23}}{S_2}\sqrt{2g(h_2 - h_3)} - V_2\frac{S_{oi}}{S_2}\sqrt{2gh_2} \\
\frac{dh_3}{dt} &= V_{13}\frac{S_{13}}{S_3}\sqrt{2g(h_1 - h_3)} + V_{23}\frac{S_{23}}{S_3}\sqrt{2g(h_2 - h_3)} - V_3\frac{S_{oi}}{S_3}\sqrt{2gh_3}
\end{align*}
\]

(4)

(5)

(6)

The equations (4), (5) and (6) are linearized around an operating point. The state space model of the system is driven as given equation (7).

\[
A = \begin{bmatrix} -C_{13} - C_1 & 0 & C_{13} \\ 0 & -C_{23} - C_2 & C_{23} \\ C_{13} & C_{23} & -C_{13} - C_{23} - C_3 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & 0 \end{bmatrix},
\]

(7)

where:

\[
C_{13} = V_{13}\frac{S_{13}}{S}\sqrt{2g(h_{10} - h_{30})},
\]

\[
C_{23} = V_{23}\frac{S_{23}}{S}\sqrt{2g(h_{20} - h_{30})},
\]

\[
C_1 = \frac{S_{10}}{S}\sqrt{2gh_{10}},
\]

\[
C_2 = \frac{S_{20}}{S}\sqrt{2gh_{20}},
\]

\[
C_3 = \frac{S_{30}}{S}\sqrt{2gh_{30}},
\]

(8)

**IV. MODEL PREDICTIVE CONTROL**

Since MPC makes use of values at specific instants of time, a continuous system with \(m\) inputs, \(q\) outputs and \(n_1\) states given by:

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t)
\]

(9)
y(t) = Cx(t)+Du(t) \tag{10}

needs to be discretized:

\[ x(k+1) = Ax(k)+Bu(k) \tag{11} \]

\[ y(k) = Cx(k)+Du(k) \tag{12} \]

where \( x \) is an input or manipulated variable and \( y \) is an output. In order to use the model in the design of discrete-time predictive control its augmented version should be found. (Matrix \( D \) is equal to zero)

Taking difference operation at both sides we get:

\[ x(k) = Ax(k-1)+Bu(k-1) \tag{13} \]

By defining \( \Delta x(k) = x(k) - x(k-1) \) and

\[ \Delta u(k) = u(k) - u(k-1) \]

and subtracting from equation (13) we obtain:

\[ \Delta x(k+1) = A\Delta x(k) - B\Delta u(k) \]

If we define \( \Delta y(k+1) = y(k+1) - y(k) \), relating the output and the state variable is possible because:

\[ \Delta y(k+1) = CA\Delta x(k) + CB\Delta u(k) \tag{14} \]

Choosing a new state variable vector

\[ x(k) = [\Delta x(k) \ T \ y(k) \ T]^T \]

we have:

\[
\begin{bmatrix}
\Delta x(k+1) \\
y(k+1)
\end{bmatrix} =
\begin{bmatrix}
A & o^T \\
CA & lqxq
\end{bmatrix}
\begin{bmatrix}
\Delta x(k) \\
y(k)
\end{bmatrix} +
\begin{bmatrix}
B \\
CB
\end{bmatrix}
\Delta u(k)
\]

\[ y(k) = [o \ \ lqxq] \begin{bmatrix}
\Delta x(k) \\
y(k)
\end{bmatrix} \tag{15} \]

Where \( lqxq \) identity matrix with dimensions \( qxq \) \( o \) – zero matrix with dimensions \( qx1 \)

The prediction of state variable and output variable is calculated as the expected values of the respective variables, hence, the noise effect to the predicted values being zero. For notational simplicity, the expectation operator is omitted without confusion.

Effectively, we have

\[ Y = Fx(ki) + \varphi \Delta U \tag{16} \]

Where

\[ F = \begin{bmatrix}
CA \\
CA^2 \\
CA^3 \\
\vdots \\
CA^{Np}
\end{bmatrix} \tag{17} \]

\[ \varphi = \begin{bmatrix}
CB & 0 & 0 & \ldots & 0 \\
CAB & CB & 0 & \ldots & 0 \\
CA^2B & CAB & CB & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
CA^{Np-1}B & CA^{Np-2}B & CA^{Np-3}B & \ldots & CA^{Np-Nc}B
\end{bmatrix} \tag{18} \]

The incremental optimal control within one optimization window is given by:

\[ \Delta U = (\varphi^T \varphi + \hat{R})^{-1}(\varphi^T \hat{R}s \ r(ki) - \varphi^T F x(ki)) \tag{19} \]

where matrix \( \varphi^T \varphi \) has dimension \( mNc \times mNc \) and \( \varphi^T F \) has dimension \( mNc \times n \), and \( \varphi^T \hat{R}s \) equals the last \( q \) columns of \( \varphi^T F \). The weight matrix \( \hat{R} \) is a block matrix with \( m \) blocks and has its dimension equal to the dimension of \( \varphi^T \varphi \).

The set-point signal is \( r(ki) = [r1(ki) \ r2(ki) \ldots \ r(q(ki))]^T \) as the \( q \) set-point signals to the multi-output system.

Applying the receding horizon control principle, the first \( m \) elements in \( \Delta U \) are taken to form the incremental optimal control:

\[ \Delta u(ki) = \begin{bmatrix}
\text{Im} & \text{Om} \\
\text{Om} & \ldots & \text{Om}
\end{bmatrix}
(\varphi^T \varphi + \hat{R})^{-1}(\varphi^T \hat{R}s \ r(ki) - \varphi^T F x(ki)) \\
= Ky r(ki) - Kmpc x(ki) \tag{20} \]

where \( \text{Im} \) and \( \text{Om} \) are, respectively, the identity and zero matrix with dimension \( m \times m \).

The control signal is \( \Delta u(ki) \) and the matrices \( (A,B,C,D) \) come from the augmented model used for the predictive control design.

With the information of \( xref(ki) \) replacing \( x(ki) \), the predictive control law is then modified to find \( \Delta U \) by minimizing:

\[ J = (Rs - Fxref(ki))^T \hat{R}s \ (Rs - Fxref(ki)) - 2\Delta U^T \varphi^T \]

\[ (Rs - Fxref(ki)) + \Delta U^T (\varphi^T \varphi + \hat{R}) \Delta U \tag{21} \]

The cost function and control signal are given by

\[ J = (Rs - Fxref(ki))^T \hat{R}s \ (Rs - Fxref(ki)) - 2\Delta U^T \varphi^T (Rs - Fxref(ki)) + \Delta U^T (\varphi^T \varphi + \hat{R}) \Delta U \tag{22} \]

\[ \Delta U = (\varphi^T \varphi + \hat{R})^{-1}(\varphi^T \hat{R}s r(ki) - \varphi^T F x(ki)) \tag{23} \]
V. SIMULATION RESULTS

A. Controller output:

Fig. 4 shows the controller signal which is given to the two input pumps. The controller output dies down to zero after the process variable is reached the set point.

B. Process variables

Fig. 5 shows the level of water in each tank. As mentioned in the control problem, the level of tank 3 is maintained at 100mm. The setpoint is achieved in 2 sampling instants.

Fig. 4 controller output

Fig. 5 level of water in tank 1, tank 2 and tank 3.
REFERENCES


