The Fuzzy Ratio Prey-Predator Model


Abstract—Uncertainty in general can be in the form of numeric or non-numeric, where the latter is qualitative and the former quantitative in nature. In numerical quantities, uncertainty can be as a result of unclear information, whereby fuzzy set theory is useful. In this paper we will use the prey-predator model with the ratio-dependent functional response to explore the uncertainty in population models. It is done by assuming triangular fuzzy number as the initial states of the model to treat the lack of information in that model. The numerical simulation will be provided. The fuzzy and deterministic interactions of the model will be compared.

Keywords—Dependency problem; Fuzzy Initial Conditions; Prey-Predator; Uncertainty.

I. INTRODUCTION

The idea of uncertainty has been around for some time within the statistics domain. However, it is only for the past few years that scientists and engineers have been thinking seriously about the implications of uncertainty in their respective fields. They have come to realize that most real world phenomenon and physical experiments cannot be described fully 100%, partly because the full information basically is not available. In differential equations, uncertainty can be considered in different forms [1]. The first case is characterized by the presence of a random input term or source term. The second is when the uncertainty enters through the parameters. And finally is when the initial conditions are random. In the literature, the effect of randomness investigated in [2-9]. The authors used different random distributions and different ways to introduce the random uncertainty in the models that describe those real world phenomenon. The fuzzy set theory is another strong tool to treat the uncertainty in the models when this uncertainty is considered as a result of unclear information. de Silva et al. [10] studied the fuzzy predator-prey population model. They elaborated the classical deterministic model by means of a fuzzy rule-based system. They also studied the stability of the critical points of the Holling–Tanner model. Xu and Gertner [11] introduced an uncertainty analysis technique, the general Fourier Amplitude Sensitivity Test (FAST), to study uncertainties in transient population dynamics. They found that the general FAST is able to identify the amount of uncertainty in transient dynamics and contributions by different demographic parameters. They applied the general FAST to a mountain goat (Oreamnos americanus) matrix population model to give a clear illustration of how uncertainty analysis can be conducted for transient dynamics arising from matrix population models.

In this paper, uncertainty of the case where the initial conditions are triangular fuzzy numbers is considered in the system of the nonlinear differential equation of the ratio-dependent predator-prey model. The dependency problem that arises in the fuzzy computation is taken into account when we compute the fuzzy interaction between the preys and their predators. The simulation is provided and the effect of fuzzy initial conditions on the interaction between preys and their predators is investigated.

II. PRELIMINARY CONCEPTS

A fuzzy number is a convex fuzzy set $A$ of $\mathbb{R}$, for which the following conditions hold:

i. $A$ is normalized, i.e. $\exists x \in \mathbb{R} : \mu_{A(x)} = 1$, where $\mu_{A(x)}$ is the membership function of a fuzzy number $A$,

ii. $\mu_{A(x)}$ upper semicontinuous,

iii. $\{x \in \mathbb{R} : \mu_{A(x)} = \alpha\}$, are compact sets for $0 < \alpha \leq 1$.

We say that a fuzzy number is Trapezoidal (Triangular, Gaussian) fuzzy number if its membership function is Trapezoid (Triangular, Gaussian). The membership function of a Trapezoidal fuzzy number will be interpreted as follows:

$$
\mu_{A(x)} = \begin{cases} 
0 & x < a_1 \\
\frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\
1 & a_2 \leq x \leq a_3 \\
\frac{a_4-x}{a_4-a_3} & a_3 \leq x \leq a_4 \\
0 & x > a_4
\end{cases}
$$

(1)

(When $a_2 = a_3$, (1) represents the membership function of triangular fuzzy number.)
If \( F(\alpha) \) is the set of all fuzzy numbers, and \( A \in F(\alpha) \), we can characterize \( A \) by its \( \alpha \)-cuts by following closed bounded intervals:

\[
A^\alpha = \{ x \in \mathbb{R} : \mu_{A^\alpha} \geq \alpha \} = \left[ a^\alpha, a^\alpha \right], \quad 0 < \alpha \leq 1
\]

(2)

\[
A^0 = \{ x \in \mathbb{R} : \mu_{A^0} \geq \alpha \} = \left[ a^0, a^0 \right], \quad 0 < \alpha \leq 1
\]

(3)

Operations on fuzzy numbers can be described as follows:

If \( A, B \in F(\alpha) \) then for \( 0 < \alpha \leq 1 \)

\[
[A + B]^\alpha = \left[ a^\alpha + b^\alpha, a^\alpha + b^\alpha \right],
\]

\[
[A - B]^\alpha = \left[ a^\alpha - b^\alpha, a^\alpha - b^\alpha \right],
\]

\[
[A B]^\alpha = \left[ \min \{ a^\alpha, b^\alpha \}, \max \{ a^\alpha, b^\alpha \} \right],
\]

\[
pA = p A^\alpha \text{ where } p \text{ is scalar.}
\]

Let \( d_h : F(\alpha) \times F(\alpha) \rightarrow \mathbb{R} \),

\[
d_h(A, B) = \text{Sup max} \left\{ |a^\alpha - b^\alpha|, |a^\alpha - b^\alpha| \right\} \text{ be Hausdorff}
\]

distance between fuzzy numbers, and \( (F(\alpha), d_h) \).

Definition

Let \( I \) be a real interval. A mapping \( x : I \rightarrow F(\alpha) \) is called a fuzzy process. We denote to fuzzy process as:

\[
[x(t)]^\alpha = [x(t)^\alpha_1, x(t)^\alpha_2], \quad t \in I \text{ and } 0 < \alpha \leq 1.
\]

(4)

The fuzzy derivative of a fuzzy process \( x(t) \) as in \([12]\) is defined by:

\[
\left[ \dot{x}(t) \right]^\alpha = \left[ \dot{x}(t)^\alpha_1, \dot{x}(t)^\alpha_2 \right], \quad t \in I \text{ and } 0 < \alpha \leq 1.
\]

(5)

A. Fuzzy initial fuzzy problem (FIVP)

The initial value problem can consider as

\[
\frac{dx(t)}{dt} = f \left( t, x(t) \right); \quad x(0) = x_0,
\]

(6)

where \( f : D \rightarrow F(\alpha) \) is a continuous mapping and \( x_0 \in F(\alpha) \) with \( \alpha \)-cut intervals

\[
[x_0]^\alpha = [x_0^\alpha_1, x_0^\alpha_2], \quad 0 < \alpha \leq 1
\]

(7)

When \( x = x(t) \) is a fuzzy number, the extension principle of Zadeh leads to the following definition

\[
f \left( t, s(x) \right) = \sup \{ s(x) : s = f(t, x) \}, \quad s \in \mathbb{R}
\]

(8)

It follows that

\[
\left[ f(t, x(t)) \right]^\alpha = \left[ f^\alpha_1(t, x(t)), f^\alpha_2(t, x(t)) \right], \quad 0 < \alpha \leq 1
\]

(9)

where

\[
f^\alpha_1(t, x(t)) = \min \left\{ f(t, x(t)) : w \in \left[ x_0^\alpha_1(t), x_0^\alpha_2(t) \right] \right\}, \quad 0 < \alpha \leq 1
\]

(10)

\[
f^\alpha_2(t, x(t)) = \max \left\{ f(t, x(t)) : w \in \left[ x_0^\alpha_1(t), x_0^\alpha_2(t) \right] \right\}, \quad 0 < \alpha \leq 1
\]

(11)

III. THE RATIO PREY-PREDATOR MODEL

Use The prey-predator model with the equivalent form of the Monod functional response (type II) has the form

\[
\frac{dx}{dt} = ax \left( 1 - \frac{x}{K} \right) - bxK, \quad x(0) = x_0 \geq 0,
\]

\[
\frac{dy}{dt} = \frac{cly}{y+bHx} - dy, \quad y(0) = y_0 \geq 0.
\]

(12)

Where the parameters \( a, b, c \) and \( d \) have ecological meaning that stand for the intrinsic growth rate of prey, a total attack rate for predator, interpreted as conversion efficiency \((0 < c < 1)\), and a death rate of predator in the absence of their prey, respectively. The parameters \( K, H \) represent carrying capacity of the environment for the prey population and the handling time a predator needs to process one unit of prey, respectively. The model (12) is usually called the traditional prey-dependent predator-prey model.

Lately, there is clear evidence from biological and physiological communities on the traditional prey-dependent prey-predator model, which is verifiable in many cases, especially when food resources are low relative to predator population densities and predators have to search for that resource. In that situation, the predators have to share and compete for food. To overcome this disadvantage, the functional and numerical responses should depend on the densities of both prey and predators. Arditi and Ginzburg \([13]\) suggested a simple method to insert predator dependence into the functional response by substitute \( x \) in the functional response by the ratio \( xy \). Under these assumptions, the dynamical behaviors of so-called ratio-dependent predator-prey model can be described as follows \([14]\) :

\[
\frac{dx}{dt} = ax \left( 1 - \frac{x}{K} \right) - \frac{bxy}{y+bHx}, \quad x(0) = x_0 \geq 0,
\]

\[
\frac{dy}{dt} = \frac{cly}{y+bHx} - dy, \quad y(0) = y_0 \geq 0.
\]

(13)

Fig. 1 The behaviors of preys and predators over
is called an equilibrium point and (23) as

\[ \begin{align*} 
\frac{dX}{dt} &= aX \left(1 - \frac{X}{K}\right) - \frac{bXY}{Y + bHX}, \\
\frac{dY}{dt} &= \frac{cbXY}{Y + bHX} - dY, 
\end{align*} \]

Substituting the condition (14) into the time dependent equations (17) and (18) as one equation as follows

\[ V_x(t, h, x, y) = \begin{cases} 
X(x), & \text{if } v_x \in \text{range}(V_x) \\
0, & \text{if } v_x \notin \text{range}(V_x) 
\end{cases} \]

Using the concept of \( \alpha \)-level, we can rewrite (21) and (22) as

\[ V_1(t, h, x, y)(v) = \begin{cases} 
\sup_{x, v \in (t, h, v)} X(x), & \text{if } v \in \text{range}(V_1) \\
0, & \text{if } v \notin \text{range}(V_1) 
\end{cases} \]

\[ V_2(t, h, x, y)(v) = \begin{cases} 
\sup_{y, v \in (t, h, v)} Y(y), & \text{if } v \in \text{range}(V_2) \\
0, & \text{if } v \notin \text{range}(V_2) 
\end{cases} \]

By applying (23) in (17) and (24) in (18) we get

\[ \begin{align*} 
X_{x+1} &= X_x + \frac{1}{6} \left( M_1 + 2M_2 + 2M_3 + M_4 \right), \\
Y_{y+1} &= Y_y + \frac{1}{6} \left( N_1 + 2N_2 + 2N_3 + N_4 \right). 
\end{align*} \]

Now, we consider the right hand side expressions of equations (17) and (18) as one equation as follows

\[ V_x(x) = \begin{cases} 
X(x), & \text{if } v_x \in \text{range}(V_x) \\
0, & \text{if } v_x \notin \text{range}(V_x) 
\end{cases} \]

Where \( (19) \) and \( (20) \) can be extended in fuzzy setting by using the formulas

\[ V_1(t, h, x, y)(v) = \begin{cases} 
\sup_{x, v \in (t, h, v)} X(x), & \text{if } v \in \text{range}(V_1) \\
0, & \text{if } v \notin \text{range}(V_1) 
\end{cases} \]

\[ V_2(t, h, x, y)(v) = \begin{cases} 
\sup_{y, v \in (t, h, v)} Y(y), & \text{if } v \in \text{range}(V_2) \\
0, & \text{if } v \notin \text{range}(V_2) 
\end{cases} \]

Using the concept of \( \alpha \)-level, we can rewrite (21) and (22) as

\[ V_1(t, h, x) = \left[ \min_{x, v \in (t, h, v)} X(x), \max_{x, v \in (t, h, v)} \right] \]

\[ V_2(t, h, y) = \left[ \min_{y, v \in (t, h, v)} Y(y), \max_{y, v \in (t, h, v)} \right] \]

By applying (23) in (17) and (24) in (18) we get

\[ \begin{align*} 
X_{x+1} &= X_x + \frac{1}{6} \left( M_1 + 2M_2 + 2M_3 + M_4 \right), \\
Y_{y+1} &= Y_y + \frac{1}{6} \left( N_1 + 2N_2 + 2N_3 + N_4 \right). 
\end{align*} \]
The minimums in (27) and (28), and the maximums in (29) and (30), can be computed by using the computation approach that explained in [20].

V. THE NUMERICAL SIMULATION

In the simulation we set the initial conditions of the system (16) as fuzzy number where \(X_0 = [28 + 2\alpha, 32 - 2\alpha]\) and \(Y_0 = [13 + 2\alpha, 17 - 2\alpha]\) are the fuzzy initial of prey and the fuzzy initial of predator respectively. In this simulation, we are not interested in the quantitative aspect of the model, but the qualitative one. So any other parameter values should give the same qualitative behavior.

Fig. 3 demonstrates the behaviors of the fuzzy ratio-dependent predator-prey model (16) when the possibility degree \(\alpha = 1\) (that means the crisp solution).

Figs. 4 and 5 displays the behavior of the boundaries fuzzy intervals of prey population, when \(\alpha = 0\), \(\alpha = 0.5\) and \(\alpha = 1\) at different iterations \(s = 60, s = 1000\).

Fig. 4 A prey fuzzy behavior over time when \((h = 0.15, s = 60)\).

Fig. 5 A prey fuzzy behavior over time when \((h = 0.15, s = 1000)\).

While Figs. 6 and 7 illustrates the behavior of the boundaries fuzzy intervals of the predator population at the same possibility degrees \(\alpha\).

Fig. 6 A predator fuzzy behavior over time when \((h = 0.15, s = 60)\).

Fig. 7 A predator fuzzy behavior over time when \((h = 0.15, s = 1000)\).

Fig. 6 A predator fuzzy behavior over time when \((h = 0.15, s = 1000)\).

Three things can be generally observed from these figures

1. First, the demeanor of all boundaries follow the deterministic behavior.
2. Secondly, the demeanor of fuzzy solution of preys has non-increasing fuzzy interval, as time increases. The fuzzy interval quickly decreases and becomes very small intervals over time. That is, the uncertainty decreases and never increases as time goes on.
3. Thirdly, the diameter of the interval of the fuzzy solution of predators is not fixed over time. i.e., the uncertainty decreases and increases over time where it increases as the number of predators increases and vice versa.

Generally, equations (27)-(30) compute the fuzzy approximated solution of the fuzzy ratio-dependent predator prey model (16) which indicate that, for fixed iteration \(s\), the result is an approximated fuzzy number which describes the possibility distributions of both populations, preys and predators.
their predators at fixed time $t$. Figs. 5 present the distribution of the prey population at different iterations. The figures show that the shape of a fuzzy number changes when time goes on which is caused by the nonlinearity of the model. Furthermore, they show how the support of fuzzy number of preys decreases over time. Indeed, in a nonlinear system of ordinary differential equations, the interaction between the dependent variables can be complex. If the variables are defined with an interval, the complexity might increase.

Similarly, Figs. 6 illustrate the possibility distribution shapes of the predators at different iterations.

On the other hand, for fixed $\alpha$, the result of the computation is the lower and the upper bounds of the fuzzy solutions' behavior at the possibility $\alpha$. Figs. 5, 6 the approximation of the boundaries fuzzy solution of the prey population and predators over time, for different possibility degrees $\alpha$. The figures also show the demeanors of the fuzziness of preys and predators over time, where it is non-increasing for preys and wobbling for their predators.

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prey population while the fuzzy behavior of prey is limited by the carrying capacity $K$.

The points $e_1$'s that defined in (15) represent the equilibrium points of the ratio-dependent prey-predator model (13), where the origin $e_1$ is a singular equilibrium. It makes the model system (13) is not able directly to linearize at this state. The membership function $\mathcal{X}_{e_1}$ and $\mathcal{X}_{e_i}$ of the equilibrium points $e_2$ and $e_3$, are the equilibrium points of the fuzzy model (16) [21, 22]. Therefore, the fuzzy equilibrium points is considered stable if the system always stays near to it after small disturbances where the fuzzy initial states of the system begin at adequately small near the fuzzy equilibrium point. That means the fuzzy system is attracted and the point is a fuzzy attractor point. If the system returns and approaches the fuzzy equilibrium point, then the point is asymptotically fuzzy stable. Otherwise, it is fuzzy unstable. Here we are interested with coexistence fuzzy equilibrium $\mathcal{X}_{e_i}$. Figs. 9 show the boundaries of the fuzzy intervals of the interaction between the preys and their predators for a different possibility degree $\alpha$, and the phase plane of the fuzzy model. These figures illustrate that, this fuzzy model moves away from the interior equilibrium $\mathcal{X}_{e_1}$ which is unstable.

Fig. 10 fuzzy phase plane of predators and preys over time

VI. CONCLUSION

Often, uncertainty is a term used when we are faced with situations where we are not fully sure. In this paper, our main concern is the interplay between population models and aspects of uncertainty. For that, we have used the predator-prey model with the ratio-dependent functional response to explore the uncertainty in population models. We have only looked at the uncertainty in the initial values. Our primary goal is to pursue the challenge posed by the need of dealing with uncertainty and quantifying it. We have used the fuzzy set theory as strong and powerful tool to treat our problem. The results show that the fuzzy behavior represents the generalization of the crisp behavior and makes the description of the phenomenon more realistic than the classical one. Moreover, the interactions between the components of the models will not be classical. It will take the form of uncertainty that is limited by the fuzzy range, which is different for every possibility degree.

REFERENCES