A Mathematical Model for Controlling the Spread of Ebola Virus Disease in Nigeria


Abstract—In this research work, we develop and analyse a deterministic model for controlling the spread of Ebola Virus Disease (EVD) in a population with vital dynamics (birth and death rates are not equal), incorporating quarantining of infectious individuals which is influenced by availability of isolation centres and surveillance coverage. We also considered improved personal hygiene of the susceptible population influenced by public enlightenment campaign. Numerical simulations showed that improved personal hygiene and quarantining of infectious individuals are enough to control the spread of EVD, with improved personal hygiene being the more effective and efficient of the two control parameters.

Keywords—Effective reproduction number, Endemic, Quarantine, Vital dynamics

I. INTRODUCTION

Ebola virus disease (EVD) (formerly known as Ebola haemorrhagic fever), named after the river in Democratic Republic of Congo (DRC, formerly Zaire) where it was initially discovered in 1976, is an avirulent filovirus that is known to affect humans and primates. The virus is most commonly spread via personal contact, and it has an incubation period of two to twenty – one days. It takes approximately eight hours for the virus to replicate, and can occur several times before the onset of symptoms. "Hundreds to thousands of new virus particles are then released during periods of hours to a few days, before the cell dies." [1]. Symptoms that occur within a few days after transmission include, high fever, headache, muscle aches, stomach pain, fatigue, diarrhea sore throat, hiccups, rash, red and itchy eyes, vomiting blood, bloody diarrhea [2]. The death rate of Ebola is somewhere between 50% to 90%. Until now, there is no specific cure or vaccine for the virus[4] – [8]. The research aims to analyze the effectiveness of quarantine and improved personal hygiene as control measures.

II. MODEL FORMULATION

The total population \(N\) is divided into four (4) classes of Susceptible\(S\), Latent\(L\), Infectious\(I\) and Recovered\(R\) individuals. The model parameters are defined in Table I.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
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<tbody>
<tr>
<td>(\pi)</td>
<td>recruitment rate</td>
</tr>
<tr>
<td>(\mu)</td>
<td>death removal rate</td>
</tr>
<tr>
<td>(\beta)</td>
<td>effective contact rate with infectious individuals</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>progression rate of infected individuals to infectious individuals</td>
</tr>
<tr>
<td>(\delta)</td>
<td>death rate due to disease</td>
</tr>
<tr>
<td>(\tau_1)</td>
<td>recovery rate of infected individuals due to treatment</td>
</tr>
<tr>
<td>(\tau_2)</td>
<td>recovery rate of infectious individuals due to treatment</td>
</tr>
<tr>
<td>(q)</td>
<td>number of quarantined individuals</td>
</tr>
<tr>
<td>(a)</td>
<td>surveillance coverage</td>
</tr>
<tr>
<td>(v)</td>
<td>availability of isolation centres</td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>enhanced personal hygiene due to public enlightenment</td>
</tr>
<tr>
<td>(\phi)</td>
<td>rate of public enlightenment</td>
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</table>

The corresponding mathematical equations can be described by a system of ordinary differential equations as follows:

\[
\frac{dS}{dt} = \pi - \frac{\beta (1 - q \epsilon v)(1 - \epsilon \phi)I}{N} S - \mu S
\]  

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\[ \frac{dL}{dt} = \gamma L - k_2 L \tag{2} \]
\[ \frac{dI}{dt} = \tau_1 L + \tau_2 I - \mu R \tag{3} \]
\[ \frac{dR}{dt} = \beta(1 - q\alpha) (1 - \delta\phi) N - k_1 L \tag{4} \]

where,
\[ k_1 = (\tau_1 + \gamma + \mu) \tag{5} \]
\[ k_2 = (\tau_2 + \delta + \mu) \tag{6} \]

III. ANALYSIS OF MODEL

A. Disease-Free Equilibrium

At equilibrium, (1) - (4) are set to equal zero. That is,
\[ \frac{dS}{dt} = \frac{dL}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0 \tag{7} \]

We define \((S, L, I, R) = (S^0, L^0, I^0, R^0)\) in (1) - (4). Consequently,
\[ I^0 = \frac{\gamma L^0}{k_2} \tag{8} \]
\[ R^0 = L^0 \left( \frac{k_1 \tau_1 + \gamma \tau_2}{\mu k_2} \right) \tag{9} \]
\[ S^0 = \frac{\beta (1 - q\alpha\nu) (1 - \epsilon\phi) N^0 + \pi N^0}{L^0} = 0 \tag{10} \]

Substituting equation (11) into (8) - (10) respectively, we obtain
\[ (S^0, L^0, I^0, R^0) = \left( \frac{\pi}{\mu}, 0, 0, 0 \right) \tag{12} \]

Equation (12) is the Disease-Free Equilibrium (DFE).

B. Endemic Equilibrium

We define \((S, L, I, R) = (S^*, L^*, I^*, R^*)\) and set (1) - (4) to equal zero respectively. Thus,
\[ S^* = \frac{k_1 k_2 N^*}{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi)} \tag{13} \]
\[ L^* = \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi)}{\beta_y \gamma k_1 (1 - q\alpha\nu)(1 - \epsilon\phi) + \mu k_1 k_2 N^*} \tag{14} \]
\[ I^* = \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi)}{\beta k_1 k_2 (1 - q\alpha\nu)(1 - \epsilon\phi)} \tag{15} \]
\[ R^* = \frac{1}{\beta} \left[ \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) + \mu k_1 k_2 N^*}{\beta_y \gamma k_1 (1 - q\alpha\nu)(1 - \epsilon\phi)} \right] (\tau_1 k_2 + \tau_2 \gamma) \tag{16} \]

Therefore, at endemic equilibrium, \((S, L, I, R) = (S^*, L^*, I^*, R^*),\) given by (13) - (16).

C. Effective Reproduction Number \(R_c\) of DFE

To derive the effective reproduction number, \(R_c\) of the DFE we employ the next generation operator technique described by [10], and which was subsequently analyzed by [11] thus:
\[ R_c = \rho(K) \tag{17} \]

where \(\rho(K)\) denotes the spectral radius of the next generation matrix \(K\). The matrix \(K\) is defined by
\[ K = F \omega^{-1} \tag{18} \]

Thus,
\[ F \omega^{-1} = \begin{bmatrix} \beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) S^0 & \beta (1 - q\alpha\nu)(1 - \epsilon\phi) S^0 \\ k_1 k_2 N^0 & 0 \end{bmatrix} \tag{19} \]

and
\[ \rho(F \omega^{-1}) = |F \omega^{-1} - I| = 0 \tag{20} \]

Hence,
\[ R_c = \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) S^0}{k_1 k_2} \tag{21} \]

Since \(S^0 = N^0\) at DFE,
\[ R_c = \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) N^0}{k_1 k_2} \tag{22} \]

D. Local Stability

To establish that the DFE is locally stable, we show that our effective reproduction number, \(R_c < 1\). Using the Jacobian Matrix to linearize (1) - (4) we have
\[ J = \begin{bmatrix} -\mu & 0 & -\beta (1 - q\alpha\nu)(1 - \epsilon\phi) & 0 \\ 0 & -k_1 & \beta (1 - q\alpha\nu)(1 - \epsilon\phi) & 0 \\ 0 & \gamma & -k_2 & 0 \\ 0 & \tau_1 & \tau_2 & -\mu \end{bmatrix} \tag{23} \]

On reducing (23) to row-echelon form we obtain
\[ \begin{bmatrix} -\mu & 0 & -\beta (1 - q\alpha\nu)(1 - \epsilon\phi) & 0 \\ 0 & -k_1 & \beta (1 - q\alpha\nu)(1 - \epsilon\phi) & 0 \\ 0 & 0 & \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) - k_1 k_2}{k_1} & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \tag{24} \]

The eigenvalues of (24) are found to be
\[ \lambda_1 = -\mu \tag{25} \]
\[ \lambda_2 = -k_1 \tag{26} \]
\[ \lambda_3 = \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) - k_1 k_2}{k_1} \tag{27} \]
\[ \lambda_4 = -\mu \tag{28} \]

Equations (25) - (28) implies,
\[ \lambda_1 < 0 \tag{29} \]
\[ \lambda_2 < 0 \tag{30} \]
\[ \lambda_3 < 0 \text{ iff} \left( \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) - k_1 k_2}{k_1} < 0 \right) \tag{31} \]
\[ \lambda_4 < 0 \tag{32} \]

But,
\[ \beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) - k_1 k_2 < 0 \tag{33} \]

Since,
\[ \beta_y (1 - q\alpha\nu)(1 - \epsilon\phi) < k_1 k_2 \tag{34} \]

Dividing through (34) by \(k_1 k_2\), we have
\[ \frac{\beta_y (1 - q\alpha\nu)(1 - \epsilon\phi)}{k_1 k_2} < 1 \tag{35} \]

From (35), we conclude that \(R_c < 1\). Hence, the DFE is locally stable.
E. Global Asymptotic Stability of DFE

By employing the Lyapunov principle the DFE is globally asymptotically stable if \( P < 0 \) or \( P' \leq 0 \); where,

\[
P = \gamma L + k_1 l
\]
\[
P' = \gamma L' + k_1 l'
\]

That is,

\[
P' = \gamma \left( \frac{B(1 - qav)(1 - e\phi)}{N} - S - k_1 L \right) + k_1 \left( \gamma L - k_2 l \right)
\]
\[
P' = k_1 k_2 l \left( \frac{B(1 - qav)(1 - e\phi)}{N k_1 k_2} - S - 1 \right)
\]

Since \( \frac{S}{N} \leq \frac{S^0}{N^0} \),

\[
P' \leq k_1 k_2 l \left( \frac{\beta \gamma (1 - qav)(1 - e\phi)}{N^0 k_1 k_2} - 1 \right)
\]

But at DFE \( S^0 = N^0 \). Thus,

\[
P' \leq k_1 k_2 l \left( \frac{\beta \gamma (1 - qav)(1 - e\phi)}{k_1 k_2} - 1 \right)
\]

Therefore,

\[
P' \leq k_1 k_2 l (R_c - 1)
\]

Thus, \( P' = 0 \) when \( R_c = 1 \) and \( P' \leq 0 \) when \( R_c < 1 \); thus, by Lyapunov principle, the DFE is globally asymptotically stable.

IV. NUMERICAL SIMULATION

For the purpose of model validation, in order to ensure that the model is in agreement with reality, numerical simulation is undertaken using the data provided in Table I, and varying values of the control parameters, \( q \) and \( e \). The results are displayed in Fig. 1 – Fig. 8.

<table>
<thead>
<tr>
<th>TABLE II</th>
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<tbody>
<tr>
<td>PARAMETER VALUES</td>
</tr>
<tr>
<td>Symbol</td>
</tr>
<tr>
<td>( \pi )</td>
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<tr>
<td>( \mu )</td>
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Fig. 1 Latent, Infectious and Recovered classes when \( q = 0, e = 0 \)

Fig. 2 Susceptible class when \( q = 0, e = 0 \)

Fig. 3 Latent, Infectious and Recovered classes when \( q = 1, e = 1 \)
Fig. 1 reveals the rate at which the population becomes latently infected is fast increasing when there is no control parameter in place. In Fig. 2, the susceptible population reduces at a very high rate when no control parameters are in place. Fig. 3 shows the latent, infectious and recovered classes when both control parameters are implemented at full scale (i.e. 100%). The rate of infection can be seen to have dropped drastically. In Fig. 4 the effectiveness of improved personal hygiene over quarantine is clearly exhibited. Fig. 5 exhibits a gradual drop in the rate at which susceptible individuals becomes infected. When quarantine and improved personal hygiene are implemented at full scale (i.e. 100%) in Fig. 6, the disease is put under control and dies out soon. In Fig. 7 the growth of the susceptible population is uniform when the proportion of quarantined infectious individuals and the proportion of susceptible population that improved their

V. DISCUSSION OF RESULTS

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personal hygiene are both 100%. Fig. 8 reveals a steady drop in the effective reproduction number of the disease which proves the effectiveness of the control parameters in place.

VI. CONCLUSION

Given the results obtained from the analysis of the model, we observed that a timely implementation of the control parameters would go a long way in stemming the spread of the disease in a population that has been ravaged by EVD. While this is a good thing, we must emphasize the fact that a timely identification of an outbreak remains of paramount importance in controlling the spread of the disease.

REFERENCES